The Arrowhead Torus : a Cayley Graph on the 6-valent Grid

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Abstract: The "arrowhead torus" is a broadcast graph that we define on the 6-valent grid as a Cayley graph. We borrow the term from Mandelbrot who qualifies in that way one of the Sierpinski's famous fractal constructions. The 6-valent grid $H = (V, E)$ is generated by three families of straight lines. We adopt the isotropic orientation $S \to N$, $NE \to SW$, $NW \to SE$ and define the system of generators $S = \{s_1, s_2, s_3\}$ whose elements are the three respective translations. The multiplication on S defines a group acting on the vertices of V with a basic set of relations. The *arrowhead* is the graph of a finite group generated by superimposing a cyclic relation for each direction. The arrowhead interconnection network has several important advantages. It has a bounded valence as a grid and the highest allowed valence for a 2D regular grid. As a Cayley graph, it allows recursive constructions and divide-and-conquer schemes for information dissemination, it is also vertex-transitive hence all routers will behave in a similar way. From construction it will appear finally as a good host for embedding subvalent topologies like the usual grid.

Key-words: interconnection networks, Cayley graphs, hexavalent grid.

(Résumé : tsvp)

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Le tore en “sagette” : un graphe de Cayley sur la grille *6*-valente

Résumé : Le tore dit en “sagette” (ou encore “en pointe-de-flèche”) est un graphe de diffusion que l'on définit sur la grille *6*-valente comme graphe de Cayley. Le terme est emprunté à Mandelbrot qui qualifie de cette façon l'une des célèbres constructions fractales de Sierpinski. La grille *6*-valente $H = (V, E)$ est engendrée par trois familles de droites. On adopte l'orientation isotrope $S \to N$, $NE \to SW$, $NW \to SE$ et on définit le système de générateurs $S = \{s_1, s_2, s_3\}$ dont les éléments sont les trois translations respectives. La multiplication sur S définit un groupe opérant sur les sommets de V selon un ensemble de relations de base. La *sagette* est le graphe d'un groupe fini engendré en surimposant une relation cyclique sur chaque direction. Le réseau d'interconnexion en sagette a plusieurs avantages importants. En tant que grille, il est de valence bornée, la plus haute valence possible pour une grille 2D régulière. En tant que graphe de Cayley, il est récursivement constructible et autorise des schémas de type diviser-conquérir pour la dissémination d'information, il est de plus sommet-transitif et tous les routeurs se comporteront ainsi de la même manière. Par construction, il apparaitra finalement comme un hôte apte au plongement de topologies subvalentes comme la grille usuelle.

Mots-clé : réseaux d'interconnexion, graphes de Cayley, grille hexavalente.

1 Introduction

The choice of interconnection network topologies remains a critical subject in the design of efficient distributed memory parallel computers. Most of the performance limitations are due to the performance of the communication system. Between extreme cases such as weakly connected circular rings or strongly connected complete networks, a solution will result from a compromise in order to satisfy, as it was mentioned by Hillis [12], a set of sometimes incompatible requirements: small degree and small diameter, bounded degree and expandability, fault tolerant connectivity and efficient layout, and so forth. Moreover it should be of obvious interest for the routing system that the topology may provide symmetrical schemes for global communications. A symmetrical scheme means that all nodes will behave in a similar way and from this fact will arise a maximum simplicity in the design and processing of the system communication kernel. More precisely, symmetry in the topology means that the representative graph is provided with an algebraic group structure as it is the case for the hypercube and some other families of Cayley graphs [1]. The hypercube therein appeared as a promising topology in the past decade in addition to its property of minimum broadcast graph [17]. Unfortunately, degree increases with size, that brings on one troublesome hardware drawback with respect to the expandability requirement. On the contrary, that is not the case for the grid, nor for the torus, its vertex-transitive extension. That feature gives a renewal of interest to this ancient topology which had been the first one to be proposed for parallel computers [20] and, in spite of a rather large diameter, returns up-to-date [15]. The "arrowhead torus" is proposed here as a trade-off by which we attempt, while requiring the highest possible level of symmetry, to combine both simplicity of grids and regularity of Cayley graphs.

We focus first on a category of infinite two-dimensional grids, namely the *hexavalent* grid. The finite case is then carried out by an adequate construction of vertex-transitive torus. The usual "orthogonal" grid in which a vertex has four neighbours (i.e. *N*-*S*-*E*-*W*) is not unique. It can be shown that there exist exactly three types of regular 2D grids, characterized by their *valence* (a crystallographic term also used for regular graphs) : thus a regular grid can be either *3*-valent (like in a beehive), *4*-valent (the usual grid) or *6*-valent (for our proposal). The reader is referred to [8] for a better understanding of the terminology. Note that for a p-valent grid, the highest symmetry is obtained when $p = 6$; recall that symmetry is an important requirement for interconnection networks, as mentioned above.

Once the valence p has been fixed, there are several ways of sizing a finite grid and then there are several ways to wrap the grid as a p-valent torus. For example, a *4*-valent torus can be rectangular or square with an orthogonal wraparound as usual but any other *transitive* arrangement is possible as well (commonly, the rectangular case is known as a product of cycles $C_{k_1} \times C_{k_2}$ and when $k_1 = k_2 = k$ – the most symmetric case – the square is often viewed as a k-ary *2*-cube [5]). Let us define a "polyomino" as a connected set of square cells and call "prototile" the polyomino covering some finite *4*-valent grid: the term "transitive" merely means that the prototile must cover the whole plane without gaps or overlaps. Let us now define a "polyhexe" as a connected set of hexagonal cells; similarly, several ways are possible to define a prototile on the *6*-valent grid, composed now of polyhexes instead of polyominoes, and to wrap the enclosed grid as a *6*-valent torus. For details about polyominoes and polyhexes and for a relevant theory of tiling, the reader is referred to [10] and references therein. It is worth mentioning the family arising in various successive projects : FAIM-1 [7] or Mayfly [6], and HARTS [3] ; the topology is a *hexagonal* grid made up of concentric rings around the centroid and the wraparound scheme results from a skewed – asymmetric – tiling of the plane (see [7]). As far as we are concerned, our proposal deals with a recursively scalable torus provided with a self-similar feature. So, in order to avoid confusion with other hexavalent topologies, an accurate appellation is needed : we choose in the sequel to name this peculiar one a "*6*-valent arrowhead torus", or *arrowhead* for short. The reason is highlighted in Fig. 5. We borrow the term from Mandelbrot who qualifies in that way one of the Sierpinski's famous fractal constructions [14].

The arrowhead is the graph of a discrete group generated from an abstract definition – or "presentation" – composed of a finite set of relations. The infinite *6*-valent grid is generated by three families of straight lines. We adopt the *isotropic* orientation $S \to N$, $NE \to SW$, $NW \to SE$ and define the system of generators $S = \{s_1, s_2, s_3\}$ whose elements are the three respective translations. The multiplication on S defines a group acting on the vertices of the grid according to the presentation. In order to generate a finite torus on the infinite grid we enlarge the presentation by imposing a cyclic relation for each direction. Isotropy is carried out by applying the same relation for any generator of S. The generation will follow a recursive scheme : the smallest non-trivial cyclic group has order 2, so we decide to start the process in such a way that the new relation generates unidirectional cycles of length 2, then merely doubles their length at each step. The graph of the group is usually known as its *Cayley diagram*. It is a coloured digraph with one colour per generator. We require further that S be closed under inverses and define the arrowhead as the non-oriented version of the digraph. In the sequel, the digraph will be denoted by $\Gamma_{(n)}$ and the non-oriented arrowhead by $\mathcal{A}_n = (V_n, E_n)$.

It has been pointed out that the arrowhead results from an *isotropic* configuration of the set of generators. Reversing any generator would yield a *non-isotropic*, diamond-shaped version of *6*-valent torus. The *diamond* is closely related to a more usual interconnection topology, the k-ary *2*-cube [5] and appears as a skewed k-ary *2*-cube with a peculiar value of k and on which a diagonal direction of links has been added. Nevertheless we claim that, by construction, arrowhead and diamond belong to a same family and, moreover, the fact that they are isomorphic is conjectured. The diamond will be presented elsewhere, in a companion paper of the present one.

Section 2 describes the generating process of the arrowhead. General conventions are first set up on the infinite *6*-valent grid. We analyse the recursive generation of $\Gamma_{(n)}$ and the cases $n = 1$ and $n = 2$ are emphasized before dealing with the general case. A definition of the non-oriented version is issued afterwards and a hexagonal representation of the arrowhead is also displayed. Section 3 exhibits some basic properties regarding vertices, edges and diameter. In particular, a recursive labelling scheme to define V_n as well as a recursive connecting scheme to define E_n are relevant. Section 4 concludes upon a presumed versatility of the arrowhead and suggests that a more complete study of its topological properties have thereby to be pursued. For a relevant background about abstract definitions of groups and graphs of groups the reader can refer to [4, 9, 13, 16, 2, 11]. A tiling paradigm will sometimes help the proposal since there exists a close relationship between a torus and a periodic tiling; the reader will find the required material in [10, 19] and [8].

2 Generating the arrowhead

We first set up general metric conventions that define the infinite (*6*-valent) lattice in some convenient coordinate system and give the induced Cayley representation in terms of abstract definition of a group. Then, in order to generate a torus, we enlarge the previous definition by imposing a cyclic relation and show how the arrowhead can be recursively generated.

2.1 General conventions.

Although the construction has a combinatorial nature, the arrowhead arises from a metrical definition of the lattice. We adopt the orientation *N-NW-SW-S-SE-NE* as shown in Fig. 1. The lattice $H = (V, E)$, composed of a set V of vertices and a set E of edges connecting pairs of vertices, is generated by three families of straight lines with respective directions *N–S*, *SW–NE* and *SE–NW*. For a reason of symmetry, we provide the Euclidean plane with a hexagonal coordinate system ("HCS") with the point 000 (named O) as the origin (see [8] for detail). Vertices are labelled with indices $(h_1,\, h_2,\, h_3)$ in the HCS – the bar over an integer is a short notation meaning that it is negative –, so V is defined as:

$$V = \{(h_1,\, h_2,\, h_3) \in \mathbb{Z}^3 : h_1 + h_2 + h_3 = 0\} \tag{1}$$

and E is such that any vertex $\mathbf{h}$ is connected to the six neighbours $\mathbf{h} \pm\varepsilon_1$, $\mathbf{h} \pm\varepsilon_2$, $\mathbf{h} \pm\varepsilon_3$ where :

$$\Sigma = (\varepsilon_1, \varepsilon_2, \varepsilon_3) = ((1, 0, -1), (-1, 1, 0), (0, -1, 1)) \tag{2}$$

defines a *3*-fold system of generators in the HCS (Fig 1.a).

In addition we adopt an arbitrary isotropic orientation for the lines, say $S \to N$, $NE \to SW$ and $NW \to SE$, or *N–SW–SE* for short. Note that there exist altogether eight possible orientations, split up into both types, say "isotropic" (such as *N–SW–SE* and *S–NE–NW* by a rotation of π) and "anisotropic" (such as *N–NE–NW* plus five equivalent configurations by $j\pi/3$ $(j = 1, ..., 5)$ rotational symmetries).

The above metric conventions lead to a Cayley representation for the lattice H in Fig. 1.b with the set of generators $\mathrm{S} = \{s_1,\, s_2,\, s_3\}$ whose elements are the three corresponding translations related to Σ. The multiplication on S defines a group G acting on the vertices of V, with the set of relations

$$R_{(h)} : (s_1 s_2 s_3 = e) \ ; \qquad R_{(c)} : (s_1 s_2 s_1^{-1} s_2^{-1} = e) \tag{3}$$

where e is the identity. The relation $R_{(h)}$ means that multiplying the three generators in the *hexavalent* grid yields an elementary cycle. The *commutator* $R_{(c)}$ holds with any

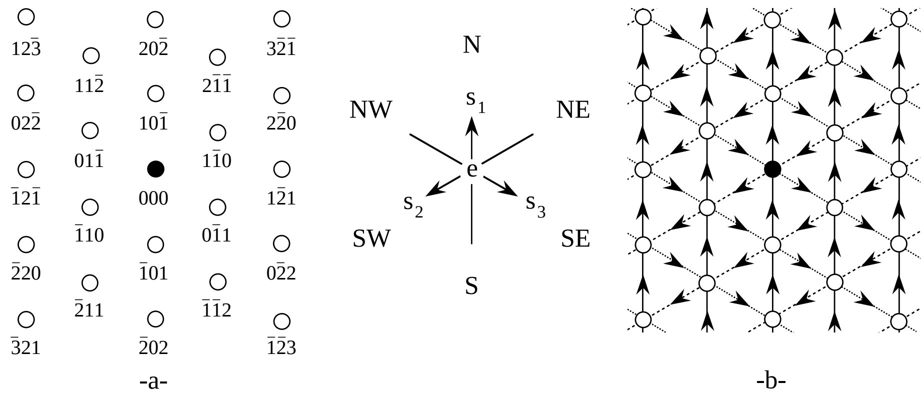

Figure 1: a/ The hexavalent grid in the hexagonal coordinate system
b/ Cayley graph of the group acting on the vertices of the grid

combination of elements of S because G is abelian. The abstract definition $\left(S; R_{(h)}, R_{(c)}\right)$ is called a *presentation* of G. It is clear that all elements of G will be of the form $s_1^{m_1} s_2^{m_2} s_3^{m_3}$ where the m_k take on integer values.

2.2 Generating $\mathcal{A}_n$.

In order to build a torus, we choose to generate unidirectional cycles of length 2^n, so we impose now a new set of cyclic relations R_n defined by

$$R_n : (s_1^{2^n} = s_2^{2^n} = s_3^{2^n} = e) \tag{4}$$

and the presentation $\left(S; R_{(h)}, R_{(c)}, R_n\right)$.

2.2.1 Generating $\mathcal{A}_1$ and $\mathcal{A}_0$.

Consider the case $n = 1$, that is : $s_1^2 = s_2^2 = s_3^2 = e$ and define the set $S_{(1)} = \{e, s_1, s_2, s_3\}$. For consistency with the following we set up $S^*_{(0)} = \{e\}$ and $S^*_{(1)} = S_{(1)}$. The presentation $(S; R_{(h)}, R_{(c)}, R_1)$ clearly defines a finite abelian group on the set $S^*_{(1)}$ displayed in the multiplication table of Fig. 2. Any binary operation is computable from the presentation. For example we have :

$$s_1 s_2 \overset{R_{(h)}}{=} s_3^{-1} \overset{R_1}{=} s_3$$

$S^*_{(1)}$	e	s_1	s_2	s_3
e	e	s_1	s_2	s_3
s_1	s_1	e	s_3	s_2
s_2	s_2	s_3	e	s_1
s_3	s_3	s_2	s_1	e

Figure 2: Multiplication table of $S^*_{(1)}$

(" $\stackrel{\rho}{=}$" means that ρ is the relevant relation). Let G_1 be the set of elements of G that now equal e as a direct consequence : those elements are of the form $s_1^{2m_1} s_2^{2m_2} s_3^{2m_3}$. We claim that G_1 is a normal subgroup of G of index *4*. Clearly G_1 is a subset of G and whenever s and t belong to G_1, then st^{-1} belongs to G_1. So G_1 is a subgroup of G. Furthermore, since G is abelian, G_1 is normal. Since G_1 is a normal subgroup of G the function $\phi_1 : G \to G/G_1$ defined by $\phi_1(s) = sG_1$ is a homomorphism. The image of ϕ_1 is the factor group

$$G/G_1 = \{G_1, s_1G_1, s_2G_1, s_3G_1\} \tag{5}$$

and its kernel is precisely G_1. The cosets of G/G_1 have elements of the form :

$$\begin{array}{rcl} G_1 & : & (s_1^{2m_1} s_2^{2m_2} s_3^{2m_3}) \\ s_1G_1 & : & (s_1^{2m_1+1} s_2^{2m_2} s_3^{2m_3}) \\ s_2G_1 & : & (s_1^{2m_1} s_2^{2m_2+1} s_3^{2m_3}) \\ s_3G_1 & : & (s_1^{2m_1} s_2^{2m_2} s_3^{2m_3+1}) \end{array}$$

and for any s', $s'' \in G$ we have $s'G_1s''G_1 = s's''G_1$. One could check that – as a consequence of the First isomorphism theorem – the table of G/G_1 and the table of $S^*_{(1)}$ display two isomorphic groups : the group of cosets and the group of their representative. It should also be pointed out that there exist exactly two groups of order 4, namely the Klein's four-group (or dihedral group D_2) and the cyclic group C_4 ; the reader can refer to the relevant literature to check that the table of $S_{(1)}$ displays a group isomorphic to D_2. Figure 3 displays the *Cayley diagram* of the group $S^*_{(1)}$ – let us call it $\Gamma_{(1)}$. Orientation *N–SW–SE* of all connections are emphasized for clarity. A vertex x stands for one element of the group and there exists one arc for each entry $x \to xs$ $(x \in S^*_{(1)}, s \in S)$ in the multiplication table.

Finally we define the *arrowhead* $\mathcal{A}_1 = (V_1 = S^*_{(1)}, E_1)$ as the non-oriented version of the digraph $\Gamma_{(1)}$.The torus $\mathcal{A}_1$ is thus a regular *2*-graph with four vertices and is readily *6*-valent. In addition we define $\mathcal{A}_0 = (V_0 = S^*_{(0)}, E_0)$ as the non-oriented version of the

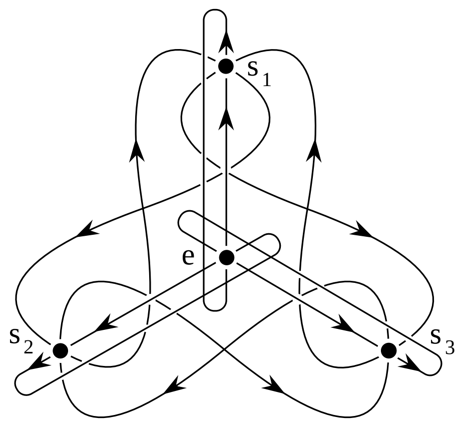

Figure 3: Graph of the group $S^*_{(1)}$

Cayley diagram $\Gamma_{(0)}$ of the trivial group defined by the presentation $(S; R_{(h)}, R_{(c)}, R_0) = (S; R_0)$.

2.2.2 Generating $\mathcal{A}_2$ from $\mathcal{A}_1$.

Although there exists a direct method, given by Coxeter & Moser [4] for computing the index of a subgroup in an abstract group by systematically enumerating the cosets, it is here more convenient to adopt a recursive scheme. We shall generate $\mathcal{A}_2$ from $\mathcal{A}_1$ then generate $\mathcal{A}_n$ in the same way for the general case. Given hence the relation $R_2 : (s_1^4 = s_2^4 = s_3^4 = e)$ and the new presentation $(S; R_{(h)}, R_{(c)}, R_2)$, let us define the set

$$S_{(2)} = \{e,\, s_1^2,\, s_2^2,\, s_3^2\}.$$

$S_{(2)}$ is a group and its multiplication table would display that $S_{(2)}$ is isomorphic to $S_{(1)}$. We should be aware that the relation R_2 replaces the relation R_1 and that R_1 is no longer fulfilled in this presentation – note incidentally that $R_1 \Rightarrow R_2$. Any operation is indeed computable from the presentation and for example we have :

$$s_1^2\, s_2^2 \overset{R_{(c)}}{=} (s_1 s_2)^2 \overset{R_{(h)}}{=} s_3^{-2} \overset{R_2}{=} s_3^2.$$

Let G_2 be the set of elements of G that equal e as a direct consequence of R_2, clearly :

$$G_2 = \{s \in G : s = s_1^{4m_1} s_2^{4m_2} s_3^{4m_3};\quad m_k \in \mathbb{Z}\}$$

and for the same reason as above, G_2 is a normal subgroup of G but it is also normal in G_1 hence G_1 is decomposable into the cosets :

$$G_2 \quad : \quad (s_1^{2(2m_1)} s_2^{2(2m_2)} s_3^{2(2m_3)})$$

$$\begin{array}{lll} s_1^2 G_2 & : & (s_1^{2(2m_1+1)} s_2^{2(2m_2)} s_3^{2(2m_3)}) \\ s_2^2 G_2 & : & (s_1^{2(2m_1)} s_2^{2(2m_2+1)} s_3^{2(2m_3)}) \\ s_3^2 G_2 & : & (s_1^{2(2m_1)} s_2^{2(2m_2)} s_3^{2(2m_3+1)}) \end{array}$$

and we obtain a factor group:

$$G_1/G_2 = \{G_2, s_1^2 G_2, s_2^2 G_2, s_3^2 G_2\} \tag{6}$$

isomorphic to $S_{(2)}$. The index of G_2 in G_1 is $[G_1 : G_2] = 4$, therefore the index of G_2 in G is : $[G : G_2] = [G : G_1][G_1 : G_2] = 16$. More precisely – refer to the Third isomorphism theorem – G_1/G_2 is a normal subgroup of G/G_2 and then :

$$(G/G_2)/(G_1/G_2) = \{G_1/G_2, (G_1/G_2)s_1, (G_1/G_2)s_2, (G_1/G_2)s_3\}. \tag{7}$$

Let us define $I = (0, 1, 2, 3)$ and consider the set of representatives :

$$S^*_{(2)} = S_{(2)} S^*_{(1)} = \{s^2_{q_1} s_{q_0} \mid q_0, q_1 \in I\}. \tag{8}$$

It should be clear from (6) and (7) – and the First isomorphism theorem – that G/G_2 is isomorphic to $S^*_{(2)}$ and we shall say that the words of $S^*_{(2)}$ have the "canonical" form for the given presentation. To carry out a full connection of the Cayley diagram $\Gamma_{(2)}$ in Fig. 4 with arcs $x \to xs$ ($x \in S^*_{(2)}$, $s \in S$), we detail the following relations

$$\begin{array}{ll} & s_\alpha^2\, s_\beta\, s_\gamma = s_\alpha (s_\alpha\, s_\beta\, s_\gamma) \overset{R_{(h)}}{=} s_\alpha \\ R' : & s_\alpha\, s_\beta \overset{R_{(h)}}{=} s_\gamma^{-1} \overset{R_2}{=} s_\gamma^3 = s_\gamma^2 s_\gamma \\ R'' : & s_\alpha^2\, s_\beta^2 \overset{R_{(c)}}{=} (s_\alpha s_\beta)^2 \overset{R_{(h)}}{=} (s_\gamma^{-1})^2 = s_\gamma^{-2} \overset{R_2}{=} s_\gamma^2 \\ & s_\alpha^3\, s_\beta = s_\alpha^2 (s_\alpha s_\beta) \overset{R'}{=} s_\alpha^2 (s_\gamma^2 s_\gamma) = (s_\alpha^2 s_\gamma^2) s_\gamma \overset{R''}{=} s_\beta^2\, s_\gamma \end{array}$$

between elements in S, where α, β, γ are all distinct. They help to transform any product xs into the canonical form of its representative (white vertices are replications of their homologue in Fig. 4). Finally we define the arrowhead $\mathcal{A}_2 = (V_2, E_2)$ as the non-oriented version of the digraph $\Gamma_{(2)}$.

2.2.3 Generating $\mathcal{A}_n$.

Theorem 1 *The presentation $(S; R_{(h)}, R_{(c)}, R_n)$ defines a group G/G_n of order 4^n isomorphic to the set of representatives:*

$$S^*_{(n)} = \{s^{2^{n-1}}_{q_{n-1}} s^{2^{n-2}}_{q_{n-2}} \dots s^2_{q_1} s_{q_0} \mid q_k \in I = (0, 1, 2, 3), (0 \le k \le n-1)\}.$$

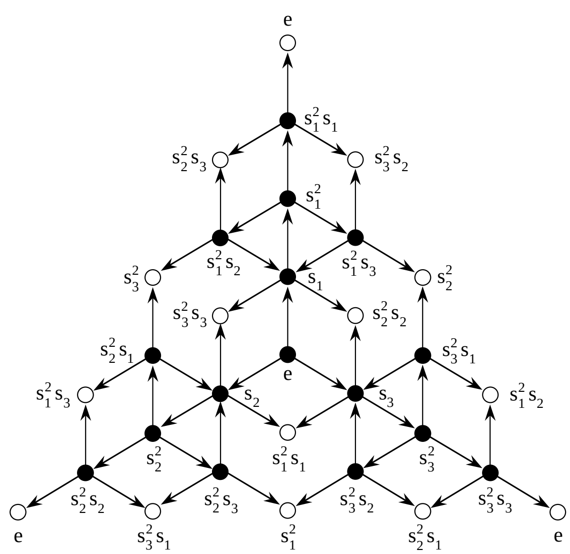

Figure 4: Graph of the group $S^*_{(2)}$

Proof Applying the Third isomorphism theorem enables an inductive proof. Define :

$$G_k = \{s \in G \mid R_k \Rightarrow (s = e)\} = \{s_1^{2^k m_1} s_2^{2^k m_2} s_3^{2^k m_3};\quad m_k \in \mathbb{Z}\}$$

$$S_{(k)} = \{e,\, s_1^{2^{k-1}},\, s_2^{2^{k-1}},\, s_3^{2^{k-1}}\}$$

for any k $(1 \leq k \leq n)$. Clearly $G_n \subset G_{n-1} \subset G$ and both G_{n-1} and G_n are normal in G, then G_{n-1}/G_n is a normal subgroup of G/G_n and the quotient $(G/G_n)/(G_{n-1}/G_n)$ is isomorphic to G/G_{n-1}. But by induction we know that $[G : G_{n-1}] = 4^{n-1}$ and that G/G_{n-1} is isomorphic to the set

$$S^*_{(n-1)} = S_{(n-1)} \ldots S_{(2)} S_{(1)}.$$

Furthermore $S_{(n)}$ is readily a group and is the set of representatives of G_{n-1}/G_n. Hence the index of G_n in G is $[G : G_n] = [G : G_{n-1}][G_{n-1} : G_n] = 4^n$ and the set :

$$S^*_{(n)} = S_{(n)} S^*_{(n-1)} = \{s's'' \in G \mid s' \in S_{(n)}, s'' \in S^*_{(n-1)}\}$$

is the set of representatives of G/G_n. □

Let us call $\Gamma_{(n)}$ the digraph of the group $S^*_{(n)}$. To carry out a full connection of $\Gamma_{(n)}$ with arcs $x \to xs$ $(x \in S^*_{(n)}, s \in S)$ we must adopt a reduction process to turn any word xs into one – and only one – canonical form $s^{2^{n-1}}_{q'_{n-1}} s^{2^{n-2}}_{q'_{n-2}} \ldots s^2_{q'_1} s_{q'_0}$ by using the relations of

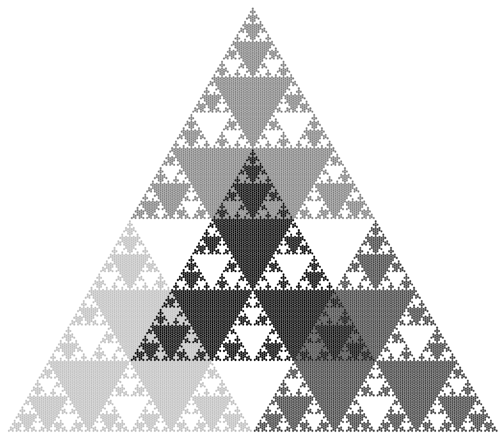

Figure 5: A view of $\mathcal{A}_7$ as a compound of $\mathcal{A}_1$ by $\mathcal{A}_6$

the presentation as it was done for $n = 2$. A convenient way is provided by the above induction which defines a *compound scheme* of $\Gamma_{(n)}$ from four "copies" of $\Gamma_{(n-1)}$, one copy corresponding with one coset of G_{n-1}/G_n. We process as follows:

1. Remove any arc in $\Gamma_{(n-1)}$ induced by the relation R_{n-1} – since R_{n-1} is no longer fulfilled – and perform the operation for each copy.

2. Replace any missing arc in $\Gamma_{(n)}$ by a new arc induced by the relation R_n.

Indeed a more general compound scheme can be settled from any subgroup of G as detailed thereafter.

Corollary 2 *For any* k $(1 \leq k \leq n-1)$, $\Gamma_{(n)}$ *can be arranged as a compound of* 4^{n-k} *copies of* $\Gamma_{(k)}$.

Proof $G_n \subset G_k \subset G$ and both G_k and G_n are normal in G. Therefore G_k/G_n is normal in G/G_n and the quotient $(G/G_n)/(G_k/G_n)$ is isomorphic to $S^*_{(k)}$ whose graph is $\Gamma_{(k)}$. The number of copies of $\Gamma_{(k)}$ is $[G_k : G_n]$, that is the index of G_n in G_k. □

As above the connecting scheme consists in removing any arc induced now by the relation R_k, in each of the 4^{n-k} copies of $\Gamma_{(k)}$; then applying R_n to replace any missing arc in $\Gamma_{(n)}$. Two typical values of k are relevant, namely $k = 1$, so $\Gamma_{(n)}$ will be arranged with 4^{n-1} copies of the graph $\Gamma_{(1)}$ displayed in Fig. 3 ; or $k = n - 1$, as in the above case highlighted for $n = 7$ in Fig. 5 – for clarity, only the metric distribution of the vertex set is drawn.

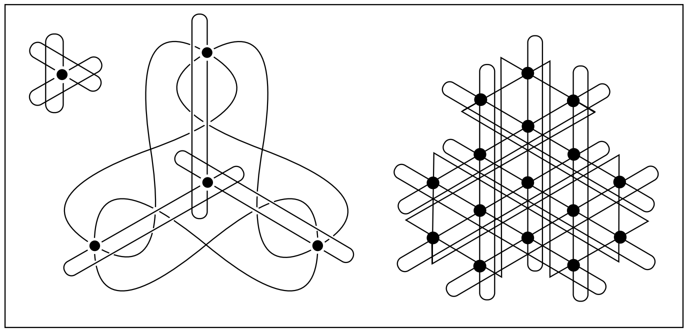

Figure 6: A view of $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$

2.3 Definition of the Arrowhead

Definition 3 *The arrowhead* $\mathcal{A}_n = (V_n, E_n)$ *is the non-oriented version of the digraph* $\Gamma_{(n)}$, *the Cayley graph of the group* $\big(S; R_{(h)}, R_{(c)}, R_n\big)$. *It has* $N = 4^n$ *vertices and* $3 \cdot 4^n$ *edges. The set of vertices is* $V_n = S^*_{(n)}$. *In addition we require that* S *be closed under inverses so that the set of edges can be defined from* $\Gamma_{(n)}$ *as* $E_n = \{\{x, y\} \in V_n \times V_n \mid x^{-1}y \in S\}$.

All definitions which will be set up in the sequel for the arrowhead result from the generating process of $\Gamma_{(n)}$. In particular, the compound scheme of $\Gamma_{(n)}$ will be applied on $\mathcal{A}_n$ as well. A complete view of $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$ is displayed in Fig. 6 ; $\mathcal{A}_2$ is shown in its hexagonal representation as described thereafter (compare with the digraph $\Gamma_{(2)}$ in Fig. 4).

2.4 Hexagonal representation of the arrowhead.

A holeless, hexagonal arrowhead is depicted in Fig. 7. It results from an adequate translation of the north, southwest, southeast holey "heads" of the original $\mathcal{A}_n$. Schematically, the "area" of $\mathcal{A}_n$ is decomposable into nine equilateral triangles as depicted in the pictogram, the boundaries being delimited by the elements of $S_{(n)}$. For simplicity we call, T_1 say, such a "head". This northern head is translated through the vector $-2^n \cdot \varepsilon_1$ (ε_1 stands here for the unit vector in the northern direction, associated with the generator s_1) to fit the U_1 area. Note that edges are not broken in the transformation. Metrically, that involves a reduction of the average length of "wrapped" edges. The fact that this transformation is allowable can be roughly explained in terms of tiling by regarding the arrowhead as a prototile which can cover the whole plane without gaps or overlaps [10, 19, 8]. Finally, since both graphs are equivalent, we call the transformed one the *hexagonal* or *folded* arrowhead $\mathcal{A}_n$, or still the *arrowhead* for short, if it is clear from context. Both representations will be expressed in terms of coordinates in the following section.

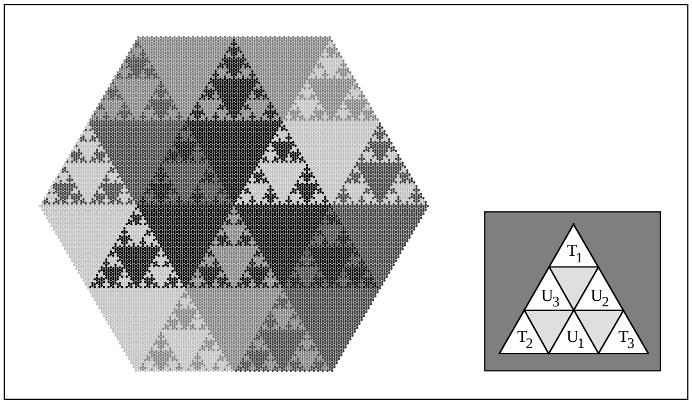

Figure 7: Hexagonal representation of $\mathcal{A}_7$

3 Basic topological properties

3.1 Vertices

3.1.1 Vertex-transitivity

Vertex-transitivity (or vertex-symmetry) is an interesting property by which all vertices have a similar behaviour [18, 2, 1]. One can easily check that, as a Cayley graph, $\mathcal{A}_n$ is vertex-transitive. In other words, for every pair of vertices there exists an automorphism of $\mathcal{A}_n$ that maps one into the other. For any fixed vertex $a \in V_n$ define the permutation $\pi_a : \sigma \rightarrow a\sigma$ ($\sigma \in V_n$) and check that adjacency is preserved by π_a. For any edge $\{x, y\} \in E_n$, the above definition of $\mathcal{A}_n$ implies $x^{-1}y \in S$. But $x^{-1}y = (ax)^{-1}(ay)$ hence $\{ax, ay\} \in E_n$ that is $\{\pi_a(x), \pi_a(y)\} \in E_n$.

3.1.2 A recursive labelling scheme

The compound schemes of $\mathcal{A}_n$ provide a straightforward quaternary notation with words of n digits $Q_n = q_{n-1}q_{n-2}\dots q_1q_0$ for numbering the elements of V_n.

1. Assume first we compose $\mathcal{A}_1$ by $\mathcal{A}_{n-1}$. Let ${}^{i}\mathcal{A}_{n-1}$ be the root, north, southwest or southeast copy of $\mathcal{A}_{n-1}$ respectively for i = *0, 1, 2* or *3*. To any vertex labelled $Q_{n-1} = q_{n-2}\dots q_1q_0$ in ${}^{i}\mathcal{A}_{n-1}$ we assign the new word $Q_n = i\,Q_{n-1}$ in $\mathcal{A}_n$.

2. Assume now we compose $\mathcal{A}_{n-1}$ by $\mathcal{A}_1$. A vertex labelled Q_{n-1} in $\mathcal{A}_{n-1}$ is replaced by the tetrad $\{Q_n = Q_{n-1}\,i\ ;\ i \in I\}$ in $\mathcal{A}_n$.

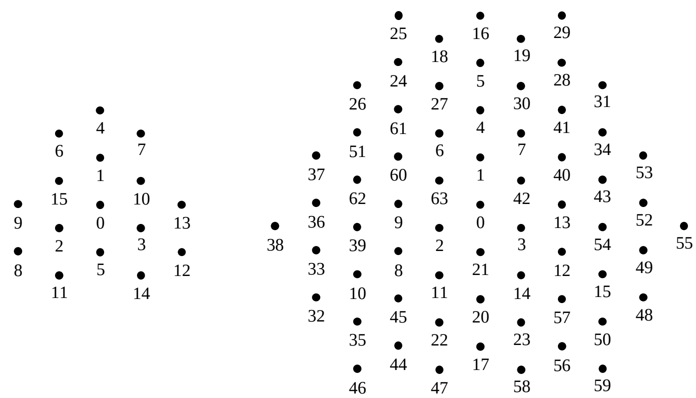

Figure 8: Labelling $\mathcal{A}_2$ and $\mathcal{A}_3$

Both schemes are consistent in the sense that the notation will be the same whatever compound alternative we choose. This combinatorial notation allows to bypass any metric notation that would have been defined on the infinite grid. An illustration of the labelling scheme is given in the hexagonal representation of Fig. 8 for $\mathcal{A}_2$ and $\mathcal{A}_3$. Let us now define in V_n the subset :

$$\mathbf{4}^k\, V_n = \{x \in V_n \,:\; x \equiv 0 \;(mod\; 4^k)\} \tag{9}$$

for any k $(0 \leq k \leq n)$. Clearly we have : $\mid \mathbf{4}^k\, V_n \mid = 4^{n-k}$. It may be useful to partition V_n into a hierarchy of $n+1$ members :

$$V_n = \{0\} \cup \bigcup_{p=1}^{n} \left(\mathbf{4}^{n-p}\, V_n - \mathbf{4}^{n-p+1}\, V_n\right) \tag{10}$$

and to regard any p–member $(p \geq 1)$ as a "descendant" of the $(p-1)$–member.

3.1.3 Labelling vertices in the hexagonal coordinate system

It will be sometimes convenient to express V_n in the hexagonal coordinate system of Sect. 2.1. Let $\Sigma = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be the system of generators defined in (2) and ε_0 the zero vector. Since an arc $x \rightarrow xs$ in $\Gamma_{(n)}$ denotes a translation in the space spanned by Σ, then the coordinates $\mathbf{h}_{Q_n} = (h_1, h_2, h_3)$ of a vertex labelled with the word Q_n in the 4–ary system satisfy

$$\mathbf{h}_{q_{n-1}q_{n-2}\cdots q_1 q_0} = \sum_{k=0}^{n-1} 2^k \varepsilon_{q_k}$$

for the *generic* representation in Fig. 5. Moreover, it would be easy to show by induction on n and using the compound scheme of $\Gamma_{(n)}$ from $\Gamma_{(n-1)}$ that the vertices of V_n are contained within an area defined by :

$$\begin{cases} h_1 - h_3 > -2^n \\ h_2 - h_1 > -2^n \\ h_3 - h_2 > -2^n \end{cases}$$

As far as the *hexagonal* arrowhead is concerned, its set of vertices can be shortly defined from (1) as :

$$\widehat{V}_n = \{(h_1,\, h_2,\, h_3) \in V : -2^n < h_1 - h_3, h_2 - h_1, h_3 - h_2 \leq 2^n\}$$

3.2 Edges

3.2.1 Edge-connectivity

The edge-connectivity λ of a graph is the minimum number of edges whose removal disconnects the graph [11]. Knowing λ involves the number of disjoint paths joining a given pair of vertices and is helpful for finding disjoint spanning trees for information dissemination.

Theorem 4 *The arrowhead $\mathcal{A}_n$ has an edge-connectivity $\lambda(\mathcal{A}_n) = 6$.*

Proof We assume the non-trivial case $n > 0$. First the valence gives a upper bound of the edge-connectivity, so $\lambda(\mathcal{A}_n) \leq 6$. We must show that $\lambda(\mathcal{A}_n) \geq 6$ and find that for any pair of vertices (x, y) there exist at least six edge-disjoint paths between x and y. There are three cycles (of length 2^n) going through any vertex and induced by the three generators of S. Let us denote $c_{x,i}$ such a cycle going through x in the direction i and $c_{y,j}$ a cycle going through y in the direction j $(i,\, j = 1, 2, 3)$. In general, all those cycles are edge-disjoint, $c_{x,i}$ and $c_{y,j}$ $(i \neq j)$ intersect at one vertex, $z_{i,j}$ say, and there exist exactly six intersections. We analyse all the paths of the form $(x, z_{i,j}, y)$ with one change of direction. For a given $z_{i,j}$ there exist four paths depending on whether we follow the direction s_i or s_i^{-1} from x to $z_{i,j}$ and the direction s_j or s_j^{-1} from $z_{i,j}$ to y. The rule is that the chosen path should not cut through another intersecting vertex. There exists one such path $(x, z_{i,j}, y)$ for a given $z_{i,j}$ leading to six edge-disjoint paths for all $z_{i,j}$, according to this rule. There is a particular case when x and y belong to a same unidirectional cycle. Without loss of generality, assume $c_{x,1} = c_{y,1}$. Therefore, $z_{2,1} = z_{3,1} = x$ whereas $z_{1,2} = z_{1,3} = y$. First there exist two unidirectional paths (x, y) following either the direction s_1 or the direction s_1^{-1}. On the other hand there exist two edge-disjoint paths $(x, z_{2,3}, y)$ either by following s_2 then s_3^{-1} or by following s_2^{-1} then s_3 and two edge-disjoint paths $(x, z_{3,2}, y)$ with a similar alternative, hence six edge-disjoint paths altogether. $\square$

3.2.2 A recursive connecting scheme

A recursive connecting scheme follows from the statement below, where $\nu_k : V_k \rightarrow V_k$ defines the neighbour, in a given direction, of any u of V_k (clearly $\nu_k \cdot \nu_k^{-1}(u) = u$).

Theorem 5 $\nu_0(0) = 0\,;\quad \nu_n(4\,x) = \nu_n^{-1}(4\,\nu_{n-1}(x)) \quad (\forall x \in V_{n-1}, n > 0)$.

Proof Observe first that $\nu_0(0) = 0$ and moreover : $\nu_1^{-1}(4\,\nu_0(0)) = \nu_1^{-1}(0) = \nu_1(0)$. Assume now we compose $\mathcal{A}_{n-1}$ by $\mathcal{A}_1$, then recall from the definition of V_n that a vertex labelled Q_{n-1} in $\mathcal{A}_{n-1}$ is replaced by the tetrad $\{Q_n = Q_{n-1}\,i\ ;\ i \in I\}$ in $\mathcal{A}_n$. In other words, let $\psi_n : V_{n-1} \to \mathbf{4}V_n$ be the one-to-one mapping defined by $\psi_n(x) = 4x$. (Since the arrowhead was initially defined on the Euclidean plane, note that it amounts to saying that ψ_n acts on V_{n-1} as a dilatation with center O and ratio 2). Hence we have also: $\psi_n(\nu_{n-1}(x)) = 4\nu_{n-1}(x)$ but: $4\nu_{n-1}(x) = \nu_n^2(4x)$ rewritten in the form : $\nu_n(4\,x) = \nu_n^{-1}(4\,\nu_{n-1}(x))$. □

The compound scheme readily defines the *N–SW–SE* connection of any vertex of $\mathbf{4}V_n$, that is, a vertex labelled $4x$ is connected to the neighbours $4x+1$, $4x+2$, $4x+3$. So the previous statement will be applied to carry out the opposite *S–NE–NW* connection. To express this fact more precisely, let X be any direction in the ordered set $(N,\ SW,\ SE)$, $(\delta_N,\ \delta_{SW},\ \delta_{SE}) = (1,2,3)$ be the set of associated increments and $\overline{X}$ the opposite direction of X in such a way that ${}^{U}\nu_k : V_k \to V_k$ can define the neighbour in the direction U of any vertex of V_k (clearly we have: ${}^{U}\nu_k^{-1} = {}^{\overline{U}}\nu_k$).

Corollary 6 *For any* $x \in V_{n-1}$ $(n > 0)$:

$$ {}^{X}\nu_n(4x) = 4x + \delta_X $$

$$ {}^{\overline{X}}\nu_n(4x) = 4\,({}^{\overline{X}}\nu_{n-1}(x)) + \delta_X $$

Proof We examine the latter case, the other is trivial. By Theorem above:

$$ {}^{\overline{X}}\nu_n(4x) = {}^{X}\nu_n(4\,({}^{\overline{X}}\nu_{n-1}(x))) $$

whence the result. □

For illustration, knowing the whole configuration of $\mathcal{A}_2$, let us examine what are the S, NE and NW neighbours of vertex 40 in $\mathcal{A}_3$ (see Fig. 8) :

$$ {}^{S}\nu_3(40) = 4\,({}^{S}\nu_2(10)) + \delta_N = 4 \cdot 3 + 1 = 13 $$

$$ {}^{NE}\nu_3(40) = 4\,({}^{NE}\nu_2(10)) + \delta_{SW} = 4 \cdot 8 + 2 = 34 $$

$$ {}^{NW}\nu_3(40) = 4\,({}^{NW}\nu_2(10)) + \delta_{SE} = 4 \cdot 1 + 3 = 7 $$

Consequently we can organize the connection from any element, $4x$ say, of $\mathbf{4}V_n$ by splitting E_n into three parts :

1. a *N–SW–SE* 3-fold connection : $4x$ is connected to the neighbours $4x+1$, $4x+2$, $4x+3$.
2. a *S–NE–NW* 3-fold reversed connection : $4x$ is therefore connected to the neighbours one can yield from the above recurrence.

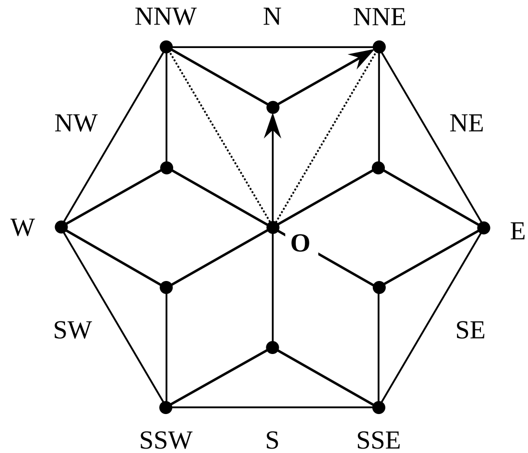

Figure 9: Centroid and antipodal points

3. a *6*-fold ring surrounding $4x$: for example the *SW*–neighbour of the *N*–neighbour of $4x$ is the *NW*–neighbour of $4x$ and so forth.

Hence E_n is wholly defined since 4^{n-1} *12*-fold disjoint connections are thus achieved.

3.3 Diameter

From the hexagonal representation of the arrowhead, it should be clear that the centroid O has *antipodal* vertices – that is, whose distance is the diameter – at the six "corners" of the hexagon, located at points NNW, W, SSW, SSE, E, NNE shown in Fig. 9. A shortest path (or "geodesic") from the centroid O to any antipodal vertex must follow only *two* directions among the six ones allowed by the three generators and their inverses. A rough estimation of the diameter can thus be given through a geometric approach : to reach for example the NNE vertex, first follow the northern direction until the barycentre of the (equilateral) triangle O–NNW–NNE, then follow the northeastern direction until destination. Since a median is half a unidirectional cycle of length 2^n, thus the diameter is close to $2 \cdot \frac{2}{3} \cdot 2^{n-1} = \frac{2}{3}\sqrt{N}$. However, a thorough computation of the diameter will take another path composed of convenient subpaths following the recursive scheme below. Without loss of generality, we can choose both antipodal vertices of the example since the arrowhead is vertex-transitive. In the sequel, ω_k $(0 \leq k \leq n)$ stands for the centroid of a subarrowhead $\mathcal{A}_k$, α stands for one NNE antipodal vertex of ω_n in $\mathcal{A}_n$ – we do not claim that α is unique – and σ_k is a word expressed in terms of generators which defines in $\mathcal{A}_k$ a geodesic from ω_k to α.

Lemma 7 *A geodesic σ_n from ω_n to α satisfies the recurrence relation :*

$$\sigma_0 = e \qquad \sigma_1 = s_1$$

$$\sigma_n = s_1^{2^{n-2}} s_2^{-2^{n-2}} \sigma_{n-2}.$$

Proof The proof is obvious for $n < 2$ so we focus on the general case. We construct a path from successive subpaths $[\omega_n, \omega_{n-1}]$, $[\omega_{n-1}, \omega_{n-2}]$, ..., $[\omega_1, \omega_0]$, where $[\omega_k, \omega_{k-1}]$ denotes a unidirectional subpath of length 2^{k-1} from centroid to centroid, following a sequence of orientations alternating from north to southeast and from southeast to north, in such a way that the subarrowhead $\mathcal{A}_k$ whose ω_k is the centroid still includes the vertex α (instead of the hexagonal representation of $\mathcal{A}_n$, the reader may find the Sierpinski-like one more convenient for illustrating the proof). In terms of generators that yields the recurrence :

$$\sigma_n = s_1^{2^{n-1}} s_3^{2^{n-2}} \sigma_{n-2}.$$

Since $s_1 s_3 = s_2^{-1}$, then rearranging the terms from the presentation we obtain the irreducible relation on the geodesic :

$$\sigma_n = s_1^{2^{n-2}} s_2^{-2^{n-2}} \sigma_{n-2}.$$

$\square$

Lemma 8 *The diameter of $\mathcal{A}_n$ satisfies the recurrence relation :*

$$D_0 = 0 \qquad D_1 = 1$$

$$D_n = D_{n-2} + 2^{n-1}.$$

Proof The diameter is the length of the geodesic. So, as a result of Lemma 7 we have :

$$D_0 = |\sigma_0| \qquad D_1 = |\sigma_1|$$

$$D_n = |\sigma_n| = |\sigma_{n-2}| + 2 \cdot 2^{n-2} = D_{n-2} + 2^{n-1}.$$

$\square$

Theorem 9 *The arrowhead $\mathcal{A}_n$ has a diameter*

$$D_n = \frac{2}{3}\left[\sqrt{N} - 1\right] \quad \text{or} \quad D_n = \frac{2}{3}\left[\sqrt{N} + 1\right] - 1 \tag{11}$$

depending on whether n *is even or odd.*

Proof Assume n even. Then from Lemma 8:

$$D_{2k} = D_{2k-2} + 2^{2k-1} \qquad (1 \leq k \leq \frac{n}{2})$$

By summation and elimination of intermediate terms we obtain :

$$D_n = \sum_{k=0}^{\frac{n}{2}-1} 4^k = \frac{2}{3}\left[2^n - 1\right]$$

(note that the term within brackets is divisible by 3).

Assume now n odd. Then from Lemma 8:

$$D_{2k+1} = D_{2k-1} + 2^{2k} \qquad (1 \leq k \leq \frac{n-1}{2})$$

In the same way we obtain:

$$D_n = \sum_{k=0}^{\frac{n-1}{2}} 4^k = \frac{2^{n+1}-1}{3} = \frac{2}{3}\,[2^n+1] - 1$$

after some little arrangement (the term within brackets is also divisible by 3). □

We close the study of the diameter with two useful relations:

Corollary 10 $D_0 = 0$ *and for any* n > 0:

- $D_{n-1} + D_n = 2^n - 1$
- $D_n = \begin{cases} 2\,D_{n-1} + 1 & if \text{ n } odd \\ 2\,D_{n-1} & if \text{ n } even \end{cases}$

Proof Immediate from the theorem above. □

4 Conclusion

This paper is a first presentation of the "arrowhead torus", a new interconnection topology generated on the hexavalent grid. Although one can notify its rather high diameter as a drawback – a diameter of grid – when compared with the hypercube, the arrowhead has a bounded valence as a grid and the highest allowable valence for a 2D regular grid. As a Cayley graph, the torus allows recursive constructions and should provide efficient divide-and-conquer schemes for information dissemination. It is also vertex-transitive hence all routers will behave in a similar way. From construction it will appear finally as a good host for embedding subvalent topologies like the usual grid. We hope to have pointed out that this kind of network would have several important advantages regarding simplicity, scalability and fault-tolerance features and consequently that it may be a good candidate able to improve the present generation of grid-based networks. Thus a thorough study of its topological properties will be pursued elsewhere, in order to show how the arrowhead is versatile.

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